ISRAEL JOURNAL OF MATHEMATICS, Vol. 44, No. 2, 1983

ERGODICITY OF CYLINDER FLOWS ARISING FROM IRREGULARITIES OF DISTRIBUTION

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ABSTRACT

Let T be the mod 1 circle group, $\alpha \in T$ be irrational and $0 < \beta < 1$. Let E be the closed subgroup of $\mathbf{R}$ generated by β and 1. Define $X = T \times E$ and $T : X \to X$ by $T(x, t) = (x + \alpha, t + 1_{[0,\beta]}(x) - \beta)$. Then we have the theorem: *T is ergodic if and only if β is rational or* 1, *α and β are linearly independent over the rationals.*

1. Introduction

Consider the compact group $T = [0, 1)$, and let $\alpha \in T$ be irrational and $0 < \beta < 1$ satisfy $\beta \not\in \mathbf{Z}\alpha$, i.e. β is not a multiple of α mod 1.†† Let E be the closed subgroup of $\mathbf{R}$ generated by 1 and β. Of course $E = \mathbf{R}$ if β is irrational and $E \cong \mathbf{Z}$ otherwise. Set $X = T \times E$ and define $T : X \to X$ by

$$T(x, t) = (x + \alpha, t + 1_{[0,\beta]}(x) - \beta).$$

This is up to an immaterial change of scale the most general transformation $(x, t) \to (x, t + f(x))$, where $f(x) = a\,1_{[0,\beta]}(x) - b$ satisfies $\int_T f(x)dx = 0$ (and $a \neq 0$). In his survey of topological dynamics, Veech [6, 7] asks to determine the ergodicity of T relative to Haar measure. In this paper, the complete solution to this problem is given. Partial results were obtained by Schmidt [4] ($\alpha = (\sqrt{5} - 1)/4$, $\beta = \frac{1}{2}$) and Conze and Keane [1] (α irrational, $\beta = \frac{1}{2}$). Recently, Stewart [5], based on the work of these authors, has obtained ergodicity in the generality proposed by Veech, which is under the condition $\beta \not\in \Gamma^0(\alpha)$, where

† This paper was prepared while I was very graciously hosted by the Centro de Investigacion y Estudios Avanzados, Mexico City.

†† $\beta \in Z\alpha$ is *a priori* excluded since it implies $\sum_{i=0}^{n-1} 1_{[0,\beta]}(x + i\alpha) - n\beta$ bounded.

Received May 3, 1981 and in revised form August 22, 1982

$\Gamma^0(\alpha)$ is an uncountable set of zero measure containing numbers well approximable by multiples of α (see [6] for definition).[†] In fact, we have

THEOREM A. *T is ergodic if and only if β is rational or $1, \alpha$ and β are linearly independent over the rationals.*

The condition given is easily seen to be necessary. For let $\tilde{E} = E/\mathbf{Z}$, $\tilde{X} = \mathbf{T} \times \tilde{E}$ and $\tilde{T} : \tilde{X} \to \tilde{X}$ be defined by $\tilde{T}(x, t) = (x + \alpha, t - \beta)$. The dynamical system $(\tilde{X}, \tilde{T})$ is a factor of the dynamical system (X, T) (both with Haar measures) under the natural projection. Since ergodicity projects to factors, $\tilde{T}$ must be ergodic if T is. Thus β rational or $1, \alpha$ and β linearly independent over the rationals, which is the condition for the ergodicity of $\tilde{T}$, is necessary.

The hard direction of Theorem A is proved by reversing the argument above, i.e. deducing the ergodicity of T from that of $\tilde{T}$. This is accomplished by

THEOREM B. *Let $f \in L^2_{\mathrm{loc}}(X)$ satisfy $f(T(x, t)) = f(x, t)$ a.e. Then $f(x, t + 1) = f(x, t)$ a.e.*

By Theorem B, any non-constant T-invariant function of X defines a non-constant $\tilde{T}$-invariant function on $\tilde{X}$, from which Theorem A follows.

Set $S_n(x) = \sum_{i=0}^{n-1} 1_{[0,\beta]}(x + i\alpha)$. Theorem B also yields

COROLLARY C. *For a.e. $x \in \mathbf{T}$, $\{S_n(x) - n\beta\}_{n=1}^{\infty}$ is dense in E (assuming $\beta \notin \mathbf{Z}\alpha$ as always).*

This settles in the affirmative a question posed in [7]. It was observed by Nelson Markley that the "a.e." in the statement cannot be replaced by "every", as there necessarily exist x's with $S_n(x) - n\beta$ semi-bounded.

The paper is organized as follows. The proof of the main result, Theorem B, in light of the previous discussion, is carried out in Section 2. To conclude, Section 3 contains the few lines needed to derive Corollary C.

2. Invariant functions are periodic

Our aim in this section is to prove Theorem B. Set

$$t_n(x) = S_n(x) - n\beta = \sum_{i=0}^{n-1} 1_{[0,\beta]}(x + i\alpha) - n\beta.$$

[†] None of these papers had appeared as of the writing of this paper. As the preprints were also unavailable to me, no reference to notation or results therein was made. Recently, Mark Stewart sent me a preprint of [5]. On the basis of this paper, it seems that the general idea of using periods (also called essential values, after Schmidt) is common to the various approaches, but they limit the number-theoretical demonstration of their existence to special cases (where, in the terminology of Section 2, multipliers are unnecessary).

Then $T^n(x,t)=(x+n\alpha, t+t_n(x))$. For $x\in\mathbf{R}$ let $[x]$ denote the closest integer to x, $\langle x\rangle = x-[x]$ and $\|x\|=|\langle x\rangle|$. The greatest integer in x will be written as $\lfloor x\rfloor$.

We start with a straightforward bit of measure theory showing how the orbit structure of (X,T) might yield periods:

PROPOSITION 1. *If there exist $n_k\in\mathbf{N}$ and $A_{n_k}\subset \boldsymbol{T}$, $k=1,2,\cdots$ such that $t_{n_k}(\cdot)$ is constant on A_{n_k}, $\lim_{k\to\infty} t_{n_k}(A_{n_k})=c$, $\inf_k m(A_{n_k})>0$ and $\lim_{k\to\infty}\|n_k\alpha\|=0$, then $f(x,y+c)=f(x,y)$ a.e. for any T-invariant $f\in l^2_{\mathrm{loc}}(X)$.*

PROOF. Set $t_{n_k}=t_{n_k}(A_{n_k})$, and let $f\in L^2_{\mathrm{loc}}(X)$ be T-invariant. Then for every $N>0$,

$$\lim_{k\to\infty}\int_0^1\int_N^N |f(x+n_k\alpha, y+t_{n_k})-f(x,y+c)|\,dydx=0.$$

Since convergence in L^1 implies convergence a.e. for subsequences, by diagonalization we can find a subsequence $\{n'_k\}_{k=1}^\infty\subset\{n_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} f(x+n'_k\alpha, y+t_{n'_k})=f(x,y+c)$ a.e. Then for a.e. $x\in A=\lim_{k\to\infty}\sup A_{n'_k}$,

$$f(x,y)=f(x+m_k\alpha, y+t_{m_k})\xrightarrow[k\to\infty]{} f(x,y+c)$$

for a.e. $y\in E$, where $\{m_k\}_{k-1}^\infty\subset\{n'_k\}_{k=1}^\infty$ is such that $x\in A_{m_k}$ $\forall k$. But $m(A)>0$, since $\inf_k m(A_{n'_k})>0$. Furthermore, the property of x that $f(x,y)=f(x,y+c)$ for a.e. $y\in E$ is invariant under the ergodic transformation $x\to x+\alpha$. Thus $f(x,y)=f(x,y+c)$ a.e. as desired. □

Spurred by the proposition, we define a *period approximating sequence* as a sequence $\{(n_l,A_l)\}_{l=1}^\infty$, where $t_{n_l}(\cdot)$ is constant on $A_l\subset \boldsymbol{T}$, $\lim_{l\to\infty} t_{n_l}(A_l)=c\in E$, $\inf_l m(A_l)>0$ and $\lim_{l\to\infty}\|n_l\alpha\|=0$. c will be called the period of the sequence.

A *period* of (X,T) in general is any $d\in E$ such that $f(x,y+d)=f(x,y)$ a.e. for any T-invariant $f\in L^2_{\mathrm{loc}}(X)$. Clearly the set of periods forms a closed subgroup of E.

The basic fact that allows us to get $t_{n_l}(A_l)$ converging for a period approximating sequence is its *a priori* boundedness for certain n_l:

LEMMA 2. *If $p,q>0$ satisfy $|\alpha-p/q|<1/q^2$ and $(p,q)=1$, then $|t_q(x)|<2$ $\forall x\in T$.*

This lemma is equivalent to the Denjoy–Koksma inequality. For completeness we present:

PROOF. Set $r=\alpha-p/q$. Let $0\leqq i\leqq q-1$. Then $jp=i \pmod q$ for some $0\leqq j\leqq q-1$. Thus

$$j\alpha=\frac{jp}{q}+jr=\frac{i}{q}+jr \qquad (\mathrm{mod}\ 1).$$

But $|jr| = j|r| < q \cdot 1/q^2 = 1/q$, so either $j\alpha \in (i/q, (i+1)/q)$ or $j\alpha \in ((i-1)/q, i/q)$, depending on the sign of r.

Thus every interval of the form $(i/q, (i+1)/q)$ contains exactly one of the points $j\alpha$ for $0 \leqq i, j \leqq q-1$. The desired conclusion follows immediately. □

The uniform distribution of $0, \alpha, \cdots, (q-1)\alpha$ in the lemma is well known, cf. [2].

In view of Lemma 2, we are led to consider $D(\alpha) = \{Q_n \mid P_n/Q_n$ is a partial convergent of $\alpha\}$; i.e., $P_n/Q_n = a_0 + 1/a_1 + \cdots + 1/a_n$, where $\alpha = a_0 + 1/a_1 + \cdots$. Basic facts regarding continued fractions will be assumed throughout this section. Khinchin's book [3], among many other texts, would provide more than ample background.

The following concept will be the building block with which period approximating sequences will be constructed: A *partial quotient configuration* (pqc) is a quadruple $(q, j, \delta, \varepsilon)$, where $q \in D(\alpha)$, $j \in \mathbf{Z}$ and $|j| < q$, $\delta/q = \langle q\alpha \rangle$ and $\varepsilon/q = \langle \beta - j\alpha \rangle$. Always $|\delta| < 1$. As for $|\varepsilon|$, it follows from the proof of Lemma 2 that $\|\beta - j'\alpha\| < 1/q$ for some $0 \leqq j' \leqq q-1$. Thus there always exists a $|j| < q$ for which $|\varepsilon| < 1$. In fact, we can do better. For let

$$\varepsilon(q) = q \cdot \min\{\|\beta - j\alpha\| \mid |j| < q\}$$

for $q \in D(\alpha)$. Then if $\varepsilon(q) \not\to 0$, we can already prove Theorem B:

PROPOSITION 3. *If*

$$\limsup_{\substack{q \to \infty \\ q \in D(\alpha)}} \varepsilon(q) > 0$$

then 1 *is a period of* (X, T).

PROOF. Let $\{q_n\}_{n=1}^{\infty} \subset D(\alpha)$ be such that $\varepsilon(q_n) > \delta > 0$ $\forall n$. Fix n. Observe that t_{q_n} is locally constant except for jump discontinuities of $+1$ at $0, -\alpha, \cdots, -(q_n - 1)\alpha$ and -1 at $\beta, \beta - \alpha, \cdots, \beta - (q_n - 1)\alpha$. Let $I_1, \cdots, I_{2q_n}$ denote the intervals of constancy in cyclic order. Since $t_{qn}(\cdot)$ takes on at most four values (Lemma 2), there exists a union of intervals, A_n, such that t_{q_n} is constant on A_n and $m(A_n) \geqq \frac{1}{4}$. Let A'_n be the union of intervals proximal on the right to those of A_n.

Noting that the distance between any two discontinuities of t_{q_n} is given by $\|(i-j)\alpha\|$ if the jumps are of the same sign and $\|\beta + (i-j)\alpha\|$ otherwise, where $0 \leqq i, j \leqq q_n - 1$, we have that $\min\{1/2q_n, \varepsilon(q_n)/q_n\}$ is a lower bound for the lengths $|I_i|$, $i = 1, \cdots, 2q_n$. Since also $|I_i| < 2/q_n$ (every interval of length $2/q_n$ must contain a $+1$ discontinuity as in Lemma 2), we have

$$\frac{|I_i|}{|I_j|} \geqq \tfrac{1}{2}\min\{\tfrac{1}{2}, \varepsilon(q_n)\}, \qquad 1 \leqq i, j \leqq 2q_n.$$

Letting $\varepsilon = \min\{\frac{1}{2}, \delta\}$, we thus have $m(A_n')\geqq \frac{1}{2}\varepsilon m(A_n)\geqq \frac{1}{8}\varepsilon$. Since t_{q_n} can take on A_n' only the values $t_{q_n}(A_n)\pm 1$, we can find $A_n''\subset A_n'$ such that t_{q_n} is constant on A_n'', $m(A_n'')\geqq \varepsilon/16$ and $t_{q_n}(A_n'')=t_{q_n}(A_n)\pm 1$. As $t_{q_n}(A_n)$ and $t_{q_n}(A_n'')$ are bounded, Proposition 1 applied to subsequences yields two periods of difference 1, and thus also 1 is a period as desired. □

I shall now try to present the intuition that is at the heart of the proof of Theorem B.

Let $q \in D(\alpha)$ be large. $\int_T |t_q(x)|\,dx$ may be assumed to be small, else arguing as above yields the result. We then try to find a multiplier $l \in \mathbf{N}$ such that upon setting $q' = lq$, $t_{q'}(\cdot)$ will be close to a good period (a fixed divisor of 1, i.e. $1/m$ where $m \in \mathbf{Z}\setminus\{0\}$) on a set with measure bounded from below, and $\|q'\alpha\|$ is small. This puts us in position to apply Proposition 1. Since

$$t_{q'}(x) = t_q(x) + t_q(x+q\alpha) + \cdots + t_q(x+(l-1)q\alpha)$$

is a superposition of $t_q(\cdot)$ shifted by multiples of $q\alpha$, careful examination of the distribution of t_q will give information about that of $t_{q'}$.

We shall separate matters into two cases. Set $B = \{x \mid t_q(x) = -\langle q\beta\rangle\}$. B is the set where t_q takes its value closest to zero. Since $\int_T |t_q(x)|\,dx$ is small, $\|q\beta\|$ must also be small. Thus t_q is close to one of ± 1, ± 2 on B^c, implying that B must have measure close to 1.

The first case is when a positive fraction of the integral of $|t_q(x)|$ is achieved on B, i.e. $\int_B |t_q(x)|\,dx$ is not much smaller than $\int_T |t_q(x)|\,dx$.

We then seek a multiplier l such that $B_l = B\cap(B-q\alpha)\cap\cdots\cap(B-(l-1)q\alpha)$ has measure bounded from below, $l\langle q\beta\rangle$ is close to a fixed divisor of 1 and $\|lq\alpha\|$ is small. Since $t_{q'}(x) = l\langle q\beta\rangle$ on B_l, we are in the desired situation.

In the second case we have the opposite behavior, where most of $\int_T |t_q(x)|\,dx$ is concentrated on B^c. We then demand of l that the translates of B^c fill out a good portion of T in a disjoint manner. Then $T_{q'}(x)$ will be close to ± 1, ± 2 on a sizable subset of T. In fact, the values ± 2 will be eliminated, so $t_{q'} = \pm 1$ on this set. If we also have $\|lq\alpha\|$ small, the proof is complete.

Multipliers as above do not necessarily exist for all $q \in D(\alpha)$ sufficiently large. They will, however, exist for an infinite number of them, as we shall see.

Call a pqc $(q, j, \delta, \varepsilon)$ effective if $|\varepsilon| + |\delta| < \frac{1}{2}$. The point here is that the distribution of $t_q(\cdot)$ is easily computable for an effective pqc:

Proposition 4. *Let $(q, j, \delta, \varepsilon)$ be an effective pqc. Then setting $p = [q\beta]$, we have $\langle q\beta\rangle = q\beta - p = (j/q)\delta + \varepsilon$ and:*

(I) *If* $j \geqq 0$ *then*

$$S_q(x)=\begin{cases} p+\operatorname{sign}(\varepsilon), & x \in P_o \cup \cdots \cup P_{q-j-1}, \\ p+\operatorname{sign}(\varepsilon+\delta), & x \in P_{q-j} \cup \cdots \cup P_{q-1}, \\ p, & \textit{otherwise}, \end{cases}$$

where

$$P_i=\begin{cases} I\left(-i\alpha,\,-i\alpha+\dfrac{\varepsilon}{q}\right), & 0 \leqq i \leqq q-j-1, \\ I\left(-i\alpha,\,-i\alpha+\dfrac{\varepsilon+\delta}{q}\right), & q-j \leqq i \leqq q-1. \end{cases}$$

(II) *If* $j<0$ *then*

$$S_q(x)=\begin{cases} p+\operatorname{sign}(\varepsilon), & x \in N_0 \cup \cdots \cup N_{q-j-1}, \\ p+\operatorname{sign}(\varepsilon-\delta), & x \in N_{q-j} \cup \cdots \cup N_{q-1}, \\ p, & \textit{otherwise}, \end{cases}$$

where

$$N_i=\begin{cases} I\left(\beta-i\alpha-\dfrac{\varepsilon}{q},\,\beta-i\alpha\right), & 0 \leqq i \leqq q+j-1, \\ I\left(\beta-i\alpha-\dfrac{(\varepsilon-\delta)}{q},\,\beta-i\alpha\right), & q+j \leqq i \leqq q-1. \end{cases}$$

Here

$$I(a,b)=\begin{cases} [a,b], & a \leqq b, \\ (b,a), & a>b. \end{cases}$$

PROOF. Since $q\alpha \in \mathbf{Z}+\delta/q$ and $\beta-j\alpha \in \mathbf{Z}+\varepsilon/q$, we have $q\beta-qj\alpha \in \mathbf{Z}+\varepsilon$ and $q\beta \in \mathbf{Z}+jq\alpha+\varepsilon=\mathbf{Z}+(j/q)\delta+\varepsilon$. But $|(j/q)\delta+\varepsilon|<|\delta|+|\varepsilon|<\frac{1}{2}$, so indeed $\langle q\beta\rangle=(j/q)\delta+\varepsilon$.

Now observe:

(1) $S_q(x)$ is locally constant except for jump discontinuities of $+1$ at $0,-\alpha,\cdots,-(q-1)\alpha$ and -1 at $\beta,\beta-\alpha,\cdots,\beta-(q-1)\alpha$.

(2) $S_q(x)$ is upper semi-continuous.

(3) The intervals $\{P_i\}_{i=0}^{q-1}$ or $\{N_i\}_{i=0}^{q-1}$ are disjoint.

PROOF OF (3). Suppose for instance that $j \geqq 0$ and $P_k \cap P_l \neq \emptyset$. If either $0 \leqq k,\ l \leqq q-j-1$ or $q-j \leqq k,\ l<q-1$ or $\operatorname{sign}(\varepsilon)=\operatorname{sign}(\varepsilon+\delta)$, then either $-k\alpha \in P_l$ or $-l\alpha \in P_k$, as $-k\alpha,\ -l\alpha$ would both be the left endpoints or the right endpoints of P_k, P_l. Then

$$\|(k-l)\alpha\| \leqq \max\{|P_k|,|P_l|\} \leqq \frac{|\varepsilon|+|\delta|}{q} < \frac{1}{2q}.$$

But $|k-l|<q$, so this inequality implies $k=l$ as desired. This remaining case is $0 \leqq k < q-j \leqq l \leqq q-1$ (or vice versa) and $\operatorname{sign}(\varepsilon) = -\operatorname{sign}(\varepsilon+\delta)$, which will lead to contradiction. For again either $-k\alpha \in P_l$ or $-l\alpha + (\varepsilon+\delta)/q \in P_k$. The first is impossible by the previous argument. Thus we are left with $-l\alpha + (\varepsilon+\delta)/q \in (-k\alpha, -k\alpha+\epsilon/q)$. But then

$$\|(k-l)\alpha\| < \frac{|\varepsilon|+|\delta|}{q} < \frac{1}{2q},$$

contradiction! The proof for $\{N_i\}_{i=0}^{q-1}$ is similar.

(4) The expressions given for the $S_q(x)$ have the right integrals over $\boldsymbol{T}$. For example, if $j \geqq 0$ then

$$\begin{aligned} p+(q-j)\operatorname{sign}(\varepsilon)\frac{|\varepsilon|}{q} + j\operatorname{sign}(\varepsilon+\delta)\frac{|\varepsilon+\delta|}{q} &= p + \frac{q-j}{q}\varepsilon + \frac{j}{q}(\varepsilon+\delta) \\ &= p+\varepsilon+\frac{j}{q}\delta = q\beta = \int_{\boldsymbol{T}} S_q(x)dx. \end{aligned}$$

Properties (1)–(3) yield the desired formulas for S_q up to an additive constant. Property (4) then seals the proof. □

If one tries to put the superposition idea to use, immediately surfacing is the requirement that the size of the increment in shifts, $\|q\alpha\| = |\delta|/q$, should not be much bigger than the size of the intervals $\{P_i\}_{i=0}^{q-1}$ or $\{N_i\}_{i=0}^{q-1}$ as above, which gets to be as small as $|\varepsilon|/q$. Thus we need

Lemma 5. *Let $(q, j, \delta, \varepsilon)$ be a pqc with $|\varepsilon| < |\delta|/1000$. Then there exist $q' > q$, δ' and ε' such that*

(1) *$(q', j, \delta', \varepsilon')$ is an effective pqc,*

(2) $1/100 > |\varepsilon'| > |\delta'|/1000$.

Proof. Since $|\varepsilon| < |\delta|/1000 < 1/1000$, we can define

$$\bar{q} = \max\{d \in D(\alpha) \mid d < q/1000|\varepsilon|\}.$$

Writing $r^+ = \min\{d \in D(\alpha) \mid d > r\}$ for every $r \in D(\alpha)$, we have $\bar{q}^+ \geqq q/1000|\varepsilon|$ or $1/\bar{q}^+ \leqq 1000|\varepsilon|/q$. But $\|\bar{q}\alpha\| < 1/\bar{q}^+$, so

$$\|\bar{q}\alpha\| \leqq 1000|\varepsilon|/q.$$

Now by Hurwitz's Theorem, there exists

$$q' = \min\{d \in D(\alpha) \mid d \geqq \bar{q} \text{ and } \|d\alpha\| < 1/\sqrt{5}d\},$$

and $q' \in \{\bar{q}, \bar{q}^+, \bar{q}^{++}\}$. Immediate inspection then yields $q' < 10\bar{q}$, from which $q' < q/100 \mid \varepsilon \mid$.

Then set $\varepsilon' = (q'/q)\varepsilon$ and $\delta' = q'\langle q'\alpha \rangle$. Since $\mid \varepsilon' \mid < 1/100$ and $\beta - j\alpha \in \mathbf{Z} + \varepsilon'/q'$, $(q, j, \delta', \varepsilon')$ is a pqc. It is effective because also $\mid \delta' \mid < 1/\sqrt{5}$. Finally,

$$\| q'\alpha \| \leqq \| \bar{q}\alpha \| < \frac{1000 \mid \varepsilon \mid}{q} = \frac{1000 \mid \varepsilon' \mid}{q'}$$

so $\mid \delta' \mid < 1000 \mid \varepsilon' \mid$. □

It will be important in the use of Lemma 5 that j remains unchanged, excluding for example the trivial $j' = j - \operatorname{sign}(j)q$.

The two cases of the superposition argument can now be identified as (1) and (2) below:

LEMMA 6. *Suppose that*

$$\lim_{\substack{q \to \infty \\ q \in D(\alpha)}} \varepsilon(q) = 0,$$

and let $0 < a < 1$ *be given. Then there exist infinitely many effective pqc's* $(q, j, \delta, \varepsilon)$ *such that* $\mid \varepsilon \mid \geqq \mid \delta \mid / 1000$ *and one of the following is true*:

(1) $\| q\beta \| \geqq a \mid \varepsilon \mid$, *or*

(2) $\| q\beta \| < a \mid \varepsilon \mid$ *and* $\mid j \mid \leqq q/2$.

PROOF. Let $(q, j, \delta, \varepsilon)$ be an effective pqc, q arbitrarily large, with $\mid \varepsilon \mid < 1/100$ (effectivity can be guaranteed by demanding $\mid \delta \mid < 1/\sqrt{5}$). This is the only place where $\varepsilon(q) \to 0$ is used. Utilizing Lemma 5 if necessary, let $q' \geqq q$, ε' and δ' be such that $(q', j, \delta', \varepsilon')$ is an effective pqc and $1/100 > \mid \varepsilon' \mid \geqq \mid \delta' \mid / 1000$ (no change if originally $\mid \varepsilon \mid \geqq \mid \delta \mid / 1000$).

Now we may assume

$$\| q'\beta \| = \left| \frac{j}{q'}\delta' + \varepsilon' \right| < a \mid \varepsilon' \mid \quad \text{and} \quad \mid j \mid > \frac{q'}{2},$$

otherwise one of (1) or (2) holds. Set $j' = j - \operatorname{sign}(j)q'$ and $\varepsilon'' = \varepsilon' + \operatorname{sign}(j)\delta'$. Note that $\mid j' \mid < q'/2$. Then $\beta - j'\alpha = \beta - j\alpha + \operatorname{sign}(j)q'\alpha \in \mathbf{Z} + \varepsilon''/q'$. Since $\mid \varepsilon'' \mid < \mid \varepsilon' \mid + \mid \delta' \mid < \frac{1}{2}$, $(q', j', \delta', \varepsilon'')$ is a pqc. It is effective since $\mid \varepsilon' \mid < 1/100$, $\mid (j/q')\delta' + \varepsilon' \mid < a \mid \varepsilon' \mid$ and $\mid j \mid > q'/2$ imply that $\mid \delta' \mid < 4 \mid \varepsilon' \mid < 1/25$, so $\mid \varepsilon'' \mid + \mid \delta' \mid \leqq \mid \varepsilon' \mid + \mid \delta' \mid + \mid \delta' \mid < \frac{1}{2}$.

Again using Lemma 5 if necessary, let $q'' \geqq q'$, δ'' and ε''' be such that $(q'', j', \delta'', \varepsilon''')$ is an effective pqc satisfying $\mid \varepsilon''' \mid > \mid \delta'' \mid / 1000$. Since $\mid j' \mid < q'/2 \leqq q''/2$, either (1) or (2) must hold, finishing the proof. □

If the pqc's $(q, j, \delta, \varepsilon)$ generated by Lemma 6 will have $\limsup \varepsilon > 0$, the period 1 for (X, T) will be easily constructed (note that this is not in contradiction to $\lim_{q\to\infty, q\in D(\alpha)} \varepsilon(q) = 0$). The alternative instances of (1) and (2) are then handled by the next two propositions.

PROPOSITION 7. *Let $\{(q_n, j_n, \delta_n, \varepsilon_n)\}_{n=1}^{\infty}$ be a sequence of effective pqc's satisfying $\lim_{n\to\infty} \varepsilon_n = 0$ and $|\varepsilon_n| \geqq |\delta_n|/1000$, $\|q_n\beta\| \geqq a|\varepsilon_n|$ $\forall n$, where $0 < a < 1$ is fixed. Then for every $m > 10^5/a$ there exist indexes $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ and multipliers $\{l_k\}_{k=1}^{\infty} \subset \mathbf{N}$ for which $\{(q'_{n_k}, A_k)\}_{k=1}^{\infty}$ is a period approximating sequence with period $\pm 1/m$, where $q'_{n_k} = l_k q_{n_k}$ and $A_k = \{x \in \boldsymbol{T} \mid q_{n_k} x \in (\frac{1}{3}, \frac{2}{3})\}$.*

PROOF. First observe that $\lim_{n\to\infty} \|q_n\beta\| = 0$. This holds since $\|q_n\beta\| < |\delta_n| + |\varepsilon_n| \leqq 1001|\varepsilon_n|$ and the latter tends to zero by assumption. Fix $m > 10^5/a$, and pick $\{n_k\}_{k=1}^{\infty}$ such that $\operatorname{sign}(\langle q_{n_k}\beta\rangle)$ is constant, $|\varepsilon_{n_k}| < 10^{-5}$ and $\|q_{n_k}\beta\| < 1/m$ $\forall k$. We may without loss of generality assume that $n_k = k$. Then set $l_k = \lfloor 1/(m\|q_k\beta\|) \rfloor$.

Now for $i = 0, \cdots, q'_k - 1$,

$$\|q_k i\alpha\| \leqq i\|q_k\alpha\| = i\frac{|\delta_k|}{q_k} < l_k|\delta_k| \leqq 10^3 l_k |\varepsilon_k|$$
$$\leqq 10^3 l_k \frac{\|q_k\beta\|}{a} \leqq 10^3 \frac{1}{ma} \leq 10^{-2}.$$

Since $\|q_k\beta\| < 1/10^5$, we thus have $\forall x \in A_k$

$$\begin{aligned} S_{q'_k}(x) &= \#\{0 \leqq i < q'_k \mid x + i\alpha \in [0, \beta]\} \\ &= l_k \#\{0 \leqq i < q_k \mid x + i\alpha \in [0, \beta]\} \\ &= l_k [q_k\beta]. \end{aligned}$$

Therefore

$$t_{q'_k}(x) = S_{q'_k}(x) - q'_k\beta = l_k[q_k\beta] - l_k q_k\beta = -l_k\langle q_k\beta\rangle.$$

The convergence of $t_{q'_k}(A_k)$ now follows since $\|q_k\beta\| \to 0$ implies $l_k\|q_k\beta\| \to 1/m$, so $-l_k\langle q_k\beta\rangle \to \mp 1/m$, depending on the sign of $\langle q_k\beta\rangle$. Finally

$$\|q'_k\alpha\| = \|l_k q_k\alpha\| \leqq l_k\|q_k\alpha\| \leqq \frac{1}{m\|q_k\beta\|}\|q_k\alpha\| \leqq \frac{|\delta_k|/q_k}{ma|\varepsilon_k|} \leqq \frac{1}{q_k}.$$

Thus $\lim_{k\to\infty} \|q'_k\alpha\| = 0$ to finish the proof. □

PROPOSITION 8. *Let $\{(q_n, j_n, \delta_n, \varepsilon_n)\}_{n=1}^{\infty}$ be a sequence of effective pqc's such that $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \varepsilon_n^{-1}\|q_n\beta\| = 0$ and $|\varepsilon_n| \geqq |\delta_n|/1000$, $|j_n| \leqq q_n/2$ $\forall n$. Then* *there exist indexes* $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$, *multipliers* $\{l_k\}_{k=1}^{\infty} \subset \mathbf{N}$ *and sets* $A_k \subset \boldsymbol{T}$ *such that* $\{(q'_{n_k}, A_k)\}_{k=1}^{\infty}$ *is a period approximating sequence with period* ± 1, $q'_{n_k} = l_k q_{n_k}$.

PROOF. Pick $\{n_k\}_{k=1}^{\infty}$ such that $\operatorname{sign}(\varepsilon_{n_k})$ is constant and $5|\varepsilon_{n_k}|, |\varepsilon_{n_k}^{-1}| \, \| q_{n_k}\beta \| < 10^{-5}$ $\forall k$. We may without loss of generality assume that $n_k = k$. Then set $l_k = \lfloor 1/(10^5 \varepsilon_k) \rfloor$.

For $i = 0, 1, \cdots, q_k - 1$, let $I_i = P_i$ or $I_i = N_i$ as in Proposition 4 applied to $(q_k, j_k, \delta_k, \varepsilon_k)$. I claim that for every $0 \leqq i \leqq q_k - |j_k| - 1$, $(I_i + tq_k\alpha) \cap I_j = \varnothing$ for every $0 \leqq t \leqq l_k - 1$ and $0 \leqq j \leqq q_k - 1$, unless $t = 0$ and $j = i$. To see this note that $(j - i)\alpha \in \bar{I}_j - \bar{I}_i$. If $j \neq i$ then $|j - i| < q_k$ implies $\|(j - i)\alpha\| > 1/2q_k$. But

$$|I_j| + |I_i| < 2\frac{|\varepsilon_k| + |\delta_k|}{q_k} \leqq \frac{2}{q_k}(10^{-5} + 10^{-2}) < \frac{1}{4q_k}.$$

Thus if $tq_k\alpha \in I_j - I_i$, then $\|tq_k\alpha\| \geqq \|(j - i)\alpha\| - (|I_j| + |I_i|) > 1/4q_k$. But

$$\|tq_k\alpha\| \leqq t\|q_k\alpha\| < l_k\frac{|\delta_k|}{q_k} \leqq \frac{10^3 l_k |\varepsilon_k|}{q_k} \leqq \frac{1}{10^2 q_k},$$

contradiction! Therefore $j = i$. Now

$$\|q_k\beta\| = \left|\frac{j_k}{q_k}\delta_k + \varepsilon_k\right| < 10^{-5}|\varepsilon_k| \quad \text{and} \quad |j_k| \leqq \frac{q_k}{2}$$

imply that $|\delta_k| > \frac{3}{2}|\varepsilon_k|$, i.e., $\|q_k\alpha\| > \frac{3}{2}|I_i|$. Thus for $1 \leqq t \leqq l_k - 1$, $\frac{3}{2}|I_i| < \|tq_k\alpha\| < 1/4q_k$, implying $(I_i + tq_k\alpha) \cap I_i = \varnothing$. Thus $t = 0$ as desired.

Now set $A'_k = \bigcup_{i=0}^{q_k - |j_k| - 1} I_i$ and $A_k = \bigcup_{t=0}^{l_k - 1}(A'_k - tq_k)$. Upon setting $p_k = [q_k\beta]$, the argument above shows that for every $x \in A_k$,

$$\begin{aligned} S_{q'_k}(x) &= S_{q_k}(x) + S_{q_k}(x + q_k\alpha) + \cdots + S_{q_k}(x + (l_k - 1)q_k\alpha) \\ &= (p_k + \operatorname{sign}(\varepsilon_k)) + (l_k - 1)p_k \\ &= \operatorname{sign}(\varepsilon_k) + l_k p_k. \end{aligned}$$

Thus

$$\begin{aligned} t_{q'_k}(x) &= S_{q'_k}(x) - q'_k\beta \\ &= \operatorname{sign}(\varepsilon_k) + l_k p_k - l_k q_k \beta \\ &= \operatorname{sign}(\varepsilon_k) + l_k(p_k - q_k\beta) \\ &= \operatorname{sign}(\varepsilon_k) - l_k\langle q_k\beta\rangle. \end{aligned}$$

But

$$|l_k\langle q_k\beta\rangle| = l_k\|q_k\beta\| < |\varepsilon_k|^{-1}\|q_k\beta\|,$$

and the latter term approaches zero by hypothesis, so $\lim_{k\to\infty} t_{q_k'}(A_k) = \pm 1$ according to the sign of the ε_k's.

Since A_k was already seen to be a disjoint union of $I_i + tq_k\alpha$, $i = 0,\cdots, q_k - |j_k| - 1$, $t = 0,\cdots, l_k - 1$, all of which have measure $|\varepsilon_k|/q_k$, we also have

$$m(A_k) = l_k(q_k - |j_k|)\frac{|\varepsilon_k|}{q_k} \geqq \tfrac{1}{2} l_k |\varepsilon_k| > \tfrac{1}{2}\left(\frac{1}{10^5} - \varepsilon_k\right) > 10^{-6}.$$

Here we used the inequality $|j_k| \leqq q_k/2$. Thus $\inf_k m(A_k) > 0$. Finally,

$$\| q_k'\alpha \| = \| l_k q_k \alpha \| \leqq l_k \| q_k \alpha \| < \frac{|\delta_k|/q_k}{10^5 |\varepsilon_k|} < \frac{1}{q_k},$$

so $\lim_{k\to\infty} \| q_k'\alpha \| = 0$ and $\{(q_k', A_k)\}_{k=1}^{\infty}$ is a period approximating sequence as desired. □

Gathering loose ends we finally have

PROOF OF THEOREM B. If $\lim_{q\to\infty} \sup \varepsilon(q) > 0$ we are finished by Proposition 3. Thus we may assume to the contrary, and apply Lemma 6 repeatedly with $a = \frac{1}{2}, \frac{1}{3}, \cdots$.

Suppose first that case (1) held for infinitely many pqc's for some $a > 0$. Then Proposition 7 applies, or else there exists a sequence $\{(q_n, j_n, \delta_n, \varepsilon_n)\}_{n=1}^{\infty}$ of effective pqc's satisfying $\| q_n \beta \| \geqq a |\varepsilon_n| > \delta > 0$ $\forall n$. Assuming this second possibility, set $A_n = \bigcup_{i=0}^{q_n - |j_n| - 1} I_i$, $B_n = \bigcup_{q_n - |j_n|}^{q_n - 1} I_i$ and $C_n = T \setminus (A_n \cup B_n)$, where for $i = 0, \cdots, q_n - 1$, $I_i = P_i$ or $I_i = N_i$ as in Proposition 4 applied to $(q_n, j_n, \delta_n, \varepsilon_n)$. Then $m(C_n) > \frac{1}{2}$. On the other hand $\int_T t_{q_n}(x)dx = 0$ implies that $\sup_n m(C_n) < 1$, since $|t_{q_n}(C_n)| = \| q_n\beta \| > \delta$. Thus there exist indexes $\{n_k\}_{k=1}^{\infty}$ for which $\{(q_{n_k}, C_{n_k})\}_{k=1}^{\infty}$ and $\{(q_{n_k}, D_{n_k})\}_{k=1}^{\infty}$ are both period approximating sequences, where either $D_{n_k} = A_{n_k}$ or $D_{n_k} = B_{n_k}$ $\forall k$. But $t_{q_n}(A_n), t_{q_n}(B_n) \in \{t_{q_n}(C_n) - 1, t_{q_n}(C_n) + 1\}$ $\forall n$. Therefore the periods of these sequences differ by 1, which is thus a period of (X, T).

Now suppose that it was case (2) that held for $a = \frac{1}{2}, \frac{1}{3}, \cdots$. Then again either Proposition 8 applies, or there exists a sequence $\{(q_n, j_n, \delta_n, \varepsilon_n)\}_{n=1}^{\infty}$ of effective pqc's with $|j_n| \leqq q_n/2$ and $|\varepsilon_n| > \delta > 0$ $\forall n$. Define A_n, B_n and C_n as above. Then

$$m(A_n) = (q_n - |j_n|)\frac{|\varepsilon_n|}{q_n} \geqq \tfrac{1}{2}|\varepsilon_n| > \tfrac{1}{2}\delta,$$

and the same argument finishes the proof. □

3. $\{\sum_{i=0}^{n-1} 1_{[0,\beta]}(x+i\alpha)-n\beta\}_{n=1}^{\infty}$ is dense in E

PROOF OF COROLLARY C. Suppose to the contrary that $B=\{x\in \boldsymbol{T} \mid \{t_n(x)\}_{n=1}^{\infty}$ is not dense in $E\}$ has positive measure. Then there exist $B'\subset B$ of positive measure and an open $\phi\neq I\subset E$ such that $t_n(x)\not\in I$, $n=1,2,\cdots\ \forall x\in B'$. Let $B''=B'\times[0,\varepsilon]\subset X$ and $\phi\neq I'\subset I$ open such that $T^n(B'')\cap \boldsymbol{T}\times I'=\varnothing\ \forall n>0$. Set $A=\lim_{n\to\infty}\sup T^n(B'')$. A has positive measure since it contains

$$\lim_{\substack{q\to\infty\\ q\in D(\alpha)}} \sup T^q(B''),$$

and by Lemma 2 such $T^q(B'')$ are contained in a finite measure space (and have constant measure >0). As $TA=A$, $A=A+(0,1)$ by Theorem B. Contradiction then follows by the ergodicity of $t\to t+\beta$ on $E/\mathbf{Z}$, since $A\cap \boldsymbol{T}\times I'=\varnothing$. □

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19 HASAVYON ST.
REHOVOT, ISRAEL